SOME AXIALLY SYMMETRIC PROBLEMS OF ELASTICITY THEORY

(NOKETORYE OSESIMMETRICHNYE ZADACHI TEORII UPRUGOSTI)

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M.Ia. BELEN'KII
(Leningrad)

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In the solution of certain problems of elasticity theory dealing with axially symmetric deformations of bodies of revolution, use is made of analytic functions of a complex variable. One utilizes Weber's [1] idea on the transformation of Airy's function into the stress function of an axially symmetrically-stressed body of revolution.

1. In case the deformation is symmetric with respect to the y-axis, the components of stress and of the elastic displacement in terms of cylindrical coordinates r, θ, y, are expressed, as is well known, by the following formulas:

$$\sigma_r = \frac{\partial}{\partial y}\left[\nu\Delta\varphi - \frac{\partial^2\varphi}{\partial r^2}\right], \qquad \sigma_y = \frac{\partial}{\partial y}\left[(2-\nu)\Delta\varphi - \frac{\partial^2\varphi}{\partial y^2}\right] \tag{1.1}$$

$$\sigma_\Theta = \frac{\partial}{\partial y}\left[\nu\Delta\varphi - \frac{1}{r}\frac{\partial\varphi}{\partial r}\right], \qquad \tau_{ry} = \frac{\partial}{\partial r}\left[(1-\nu)\Delta\varphi - \frac{\partial^2\varphi}{\partial y^2}\right]$$

$$u = -\frac{1+\nu}{E}\frac{\partial^2\varphi}{\partial y\partial r}, \qquad v = \frac{1+\nu}{E}\left[(2-2\nu)\Delta\varphi - \frac{\partial^2\varphi}{\partial y^2}\right] \tag{1.2}$$

Here ν is Poisson's ratio, E is Young's modulus and u is the radial displacement.

The function $\phi(r, y)$ satisfies the biharmonic equation

$$\Delta\Delta\varphi = 0 \qquad \left(\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial y^2}\right)$$

Let us introduce the operations S_0 and S_1 by setting

$$S_0(\varphi) = \frac{1}{\pi}\int_0^\infty d\alpha \int_0^\infty \varphi(\lambda, y)\,\alpha\lambda J_0(\alpha\lambda)\cos\alpha rt d\lambda = w(rt, y) \tag{1.3}$$

$$S_1(\psi) = \frac{1}{\pi}\int_0^\infty d\alpha \int_0^\infty \psi(\lambda, y)\,\alpha\lambda J_1(\alpha\lambda)\sin\alpha rt d\lambda = w_1(rt, y) \tag{1.4}$$

Here $J_R(t)$ is a Bessel function of the first kind.

Let us find the operations S_0^{-1} and S_1^{-1} which are the inverses of S_0 and S_1. We multiply Equation (1.3) by $dt/\sqrt{1-t^2}$ and carry out an integration over the interval $(-1, 1)$. We thus obtain

$$\int_{-1}^{1} \frac{w(rt, y)\,dt}{\sqrt{1-t^2}} = \int_0^\infty d\alpha \int_0^\infty \varphi(\lambda, y) J_0(\alpha\lambda) J_0(\alpha r)\, d\lambda = \varphi(r, y) = S_0^{-1}(w)$$

The last step was accomplished with the aid of Hankel's formula under the assumption that the function $\phi(r, y)$ satisfies, for example, Dirichlet's conditions and that it is continuous in r at the point (r, y). In an analogous manner we obtain

$$\psi(r, y) = \int_{-1}^{1} \frac{tw_1(rt, y)}{\sqrt{1-t^2}}\, dt$$

Setting $rt = x$, we note that

$$\frac{\partial\varphi}{\partial r} = \int_{-1}^{1} \frac{\partial w}{\partial x} \frac{tdt}{\sqrt{1-t^2}} = -\int_{-1}^{1} \frac{\partial w}{\partial x}\, d\sqrt{1-t^2} =$$

$$= -\frac{dw}{dx}\sqrt{1-t^2}\,\Big|_{-1}^{1} + r\int_{-1}^{1} \frac{\partial^2 w}{\partial x^2}\sqrt{1-t^2}dt = r\int_{-1}^{1} \frac{\partial^2 w}{\partial x^2}\sqrt{1-t^2}\, dt$$

$$\frac{\partial^2\varphi}{\partial r^2} = \int_{-1}^{1} \frac{\partial^2 w}{\partial x^2} \frac{dt}{\sqrt{1-t^2}}$$

From this we obtain the equation

$$\triangle\varphi = \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{\partial^2\varphi}{\partial y^2} = \int_{-1}^{1} \frac{\triangle_{xy} w(rt, y)}{\sqrt{1-t^2}}\, dt \qquad \left(\triangle_{xy} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \quad (1.5)$$

This equation (1.5) shows that the operation S_0^{-1} transforms the Laplacian in the plane into the Laplacian in space (in case of axial symmetry). The operation S_0^{-1} accomplishes the inverse transformation.

Harmonic functions are transformed into harmonic functions, and biharmonic ones into biharmonic ones by means of these operations. In general, S_0 transforms the operator

$$\triangle + \sum_{R=0}^{n} C_k \frac{\partial^R}{\partial y^R}$$

into the operator

$$\triangle_{xy} + \sum_{R=0}^{n} C_R \frac{\partial^R}{\partial y^R} \qquad (C_R = \text{const})$$

Let us transform Formulas (1.2) and (1.1) with the aid of the operations S_0 and S_1. We thus obtain the following formulas:

$$\frac{E}{1+\nu} S_1(u) = -\frac{\partial^2 w}{\partial x \partial y}, \qquad \frac{E}{1+\nu} S_0(v) = (2-2\nu) \triangle_{xy} w - \frac{\partial^2 w}{\partial y^2} \tag{1.6}$$

The function $\omega(x, y)$ is biharmonic in the xy-plane and can be considered as the stress function of some planar state of stress whose components are found by means of the formulas

$$\sigma_x{}^\circ = \frac{\partial^2 w}{\partial y^2}, \qquad \sigma_y{}^\circ = \frac{\partial^2 w}{\partial x^2}, \qquad \tau_{xy}{}^\circ = -\frac{\partial^2 w}{\partial x \partial y} \tag{1.7}$$

In this notation Formulas (1.6) take on the form

$$\tau_{xy}{}^\circ = \frac{E}{1+\nu} S_1(u), \qquad (1-2\nu)(\sigma_x{}^\circ + \sigma_y{}^\circ) + \sigma_y{}^\circ = \frac{E}{1+\nu} S_0(v) \tag{1.8}$$

or in complex notation

$$(1-2\nu)(\sigma_x{}^\circ + \sigma_y{}^\circ) + \sigma_y{}^\circ - i\tau_{xy}{}^\circ = \frac{E}{1+\nu}[S_0(v) - iS_1(u)] \tag{1.9}$$

Making use of the well-known representation of stresses by means of the two analytic functions of Kolosov-Muskhelishvili [2], we obtain

$$(3-4\nu)[\Phi(z) + \overline{\Phi(z)}] + z\overline{\Phi'(z)} + \overline{\Psi(z)} = \frac{E}{1+\nu}[S_0(v) - iS_1(u)]. \tag{1.10}$$

In an analogous manner we find

$$\varphi(z) + (4\nu-2)\overline{\varphi(z)} + z\overline{\varphi'(z)} + \overline{\psi(z)} = -i\,[S_0(\sigma_y) - iS_1(\tau_{ry})] \tag{1.11}$$

$$(3-4\nu)\varphi(z) + (4-4\nu)\overline{\varphi(z)} + z\overline{\varphi'(z)} + \overline{\psi(z)} = \frac{E}{1+\nu}\left[S_1\left(\frac{\partial v}{\partial r}\right) - \frac{i}{x} S_0\left(r\frac{\partial u}{\partial r}\right)\right] \tag{1.12}$$

Here

$$\varphi(z) = \Phi'(z), \qquad \psi(z) = \Psi'(z) \qquad (z = x + iy)$$

Formulas (1.10) and (1.11) make it possible, for a certain class of problems, to reduce the solution of axially symmetric problems in elasticity theory to the solution of auxiliary problems in the planar theory of elasticity. One can solve in this manner fundamental problems (the problem with given stresses or displacements on the boundary of a region, the mixed problem in which the stresses are given on a part of the boundary and the displacements are given on the rest of the boundary) for the half-space, for a thick infinite plate, for space, for an infinite cylinder and others.

Formulas (1.10) and (1.11), or (1.11) and (1.12), can be used for the construction of the integral equations of the problem of elasticity theory of an arbitrary body of revolution if one knows the conformal mapping of an axial cross-section of the body on to a half-plane. In this case one can determine the form of the functions $\Phi(z)$ and $\Psi(z)$. For example, for the exterior of the paraboloid $r^2 = 2(y + 1)$ they take the form

$$\Phi(z) = \frac{1}{2\pi i\,(4\nu - 3)\,(2i + \sqrt{-iz})} \int_\Gamma \frac{f(\sigma)\,d\sigma}{\sqrt{\sigma}\,(\sqrt{\bar\sigma} - \sqrt{\bar z})}$$

$$\Psi(z) = (4\nu - 3)\,\Phi(z) + (2i + \sqrt{-iz})^2 i\Phi'(z) - \frac{1}{2\pi i} \int_\Gamma \frac{\overline{f(\sigma)}\,d\sigma}{\sqrt{-i\sigma z}\,(\sqrt{\sigma} - \sqrt{\bar z})}$$

Here Γ is the parabola $x^2 = 2(y + 1)$ traced out from left to right.

By satisfying the boundary conditions, we derive the integral equations for the determination of the real and imaginary parts of the function $f(\sigma)$.

2. Let us apply the presented results to the solution of the problem of an infinite elastic plate situated between the planes $y = -1$ and $y = 1$.

Suppose that on the boundary of the plate we have the known stresses

$$\sigma_r(r, 1) = f_1(r), \qquad \tau_{ry}(r, 1) = g_1(r): \qquad \sigma_r(r, -1) = f_2(r), \qquad \tau_{ry}(r, -1) = g_2(r) \tag{2.1}$$

Let us make use of Equation (1.11). The right-hand side of this equation is a known function when $y = \pm 1$; namely

$$[S_0(\sigma_y) - iS_1(\tau_{ry})]y = (-1)^{k+1} = S_0(f_k) - iS_1(g_k) \qquad (k = 1, 2)$$

Formula (1.11) thus makes it possible to reduce the problem of a plate to the boundary-value problem of the strip $|y| < 1$ in the xy-plane.

It is known that a function $\chi(z)$ which is analytic in the strip $|y| < 1$ can be represented in the form $\chi(z) = \chi_1(z) + \chi_2(z)$, where $\chi_1(z)$ is analytic in the half-plane $y < 1$, while $\chi_2(z)$ is analytic in the half-plane $y > -1$.

Formula (1.11) can therefore be represented in the form

$$\begin{gathered}\varphi_1(z) + (4\nu - 2)\,\overline{\varphi_1(z)} + (z - 1)\,\overline{\varphi_1'(z)} + \overline{\psi_1(z)} + \\ + \varphi_2(z) + (4\nu - 2)\,\overline{\varphi_2(z)} + (z + i)\,\overline{\varphi_2'(z)} + \psi_2(z) = -i[S_0(\sigma_y) - iS_1(\tau_{ry})]\end{gathered} \tag{2.2}$$

We note that the solution of the problems on the elastic half-spaces $y < 1$ and $y > -1$ can be reduced to the solution of the auxiliary problems in the half-planes $y < 1$ and $y > 1$. But these problems are easily solved. We shall seek $\phi_k(z)$ and $\psi_k(z)$ in the same form in which they are obtained in the solution of the mentioned auxiliary problems. They will have the form

$$\varphi_1(z) = -\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{p_1(\zeta) - iq_1(\zeta)}{\zeta - z + i}\,d\zeta, \qquad \varphi_2(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{p_2(\zeta) + iq_2(\zeta)}{\zeta - z - i}\,d\zeta$$

$$\overline{\psi_k(z)} = -\varphi_k(z) - (4\nu - 2)\overline{\varphi_k(z)} - [z - (-1)^{k+1}i]\,\overline{\varphi_k'(z)} \qquad (k = 1, 2)$$

Substituting $\phi_k(z)$ and $\psi_k(z)$ into (2.2) and making use of the boundary conditions (2.1) we obtain the system

$$\begin{aligned}
&p_1(t) + \frac{16}{\pi}\int_{-\infty}^{+\infty}\frac{p_2(\zeta)}{[4 + (\zeta - t)^2]^2}\,d\zeta + \frac{8}{\pi}\int_{-\infty}^{+\infty}\frac{q_2(\zeta)(\zeta - t)}{[4 + (\zeta - t^2]^2}\,d\zeta = -S_1(g_1)\\
&g_1(t) - \frac{4}{\pi}\int_{-\infty}^{+\infty}\frac{q_2(\zeta)(\zeta - t)^2}{[4 + (\zeta - t)^2]^2}\,d\zeta - \frac{8}{\pi}\int_{-\infty}^{+\infty}\frac{p_2(\zeta)(\zeta - t)}{[4 + (\zeta - t)^2]^2}\,d\zeta = -S_0(f_1)\\
&p_2(t) + \frac{16}{\pi}\int_{-\infty}^{+\infty}\frac{p_1(\zeta)}{[4 + (\zeta - t)^2]^2}\,d\zeta + \frac{8}{\pi}\int_{-\infty}^{+\infty}\frac{q_1(\zeta)(\zeta - t)}{[4 + (\zeta - t)^2]^2}\,d\zeta = -S_1(g_2)\\
&g_2(t) - \frac{4}{\pi}\int_{-\infty}^{+\infty}\frac{q_1(\zeta)(\zeta - t)^2}{[4 + (\zeta - t)^2]^2}\,d\zeta - \frac{8}{\pi}\int_{-\infty}^{+\infty}\frac{p_1(\zeta)(\zeta - t)}{[4 + (\zeta - t)^2]^2}\,d\zeta = -S_0(f_2)
\end{aligned} \tag{2.3}$$

Further developments will be omitted because they would be almost exact repetitions of [3] in which there were solved problems in the theory of elasticity for an infinite strip. With the aid of the found $p_k(\zeta)$ and $q_k(\zeta)$ one finds $\phi_k(z)$, $\psi_k(z)$ and $S_0(\sigma_r)$, $S_1(\tau_{ry})$. By inversion one finds σ_r and τ_{ry}. The remaining components are expressed in terms of $\phi_k(z)$ and $\psi_k(z)$.

The problem with given displacements on the boundary of a plate is solved by the same procedure.

The preceding results can be applied to the solution of the mixed problem for a plate. This problem can be reduced to the finding of the stresses on segments where the displacements are known. One is hereby led to a system of singular integral equations with kernels of the Cauchy type.

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